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$*$-SDYM fields and heavenly spaces: I. $*$-SDYM equations as an integrable system

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Abstract

It is shown that the self-dual Yang–Mills (SDYM) equations for the $*$-bracket Lie algebra on a heavenly space can be reduced to one equation (the *master equation*). Two hierarchies of conservation laws for this equation are constructed. Then the twistor transform and a solution to the Riemann–Hilbert problem are given.

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Introduction

It turns out that many nonlinear integrable systems are reductions of SDYM equations (see, e.g., Mason and Woodhouse (1996)). The statement that all integrable systems of mathematical physics are some reductions of SDYM equations is known as *Ward's conjecture* (Ward 1985). The twistor construction for SDYM system is in a sense inherited by the reduced system. There are, however, exceptional cases which do not fit into this scheme for a finite-dimensional structure group (Mason and Woodhouse 1996).

An extension of the Lie algebra of SDYM equations to infinite-dimensional algebra of Hamiltonian vector fields provides a description of heavenly spaces of complex general relativity (Mason and Newman 1989). The nonlinear graviton construction (Penrose 1976, Penrose and Ward 1980, Mason and Woodhouse 1996) proves that the heavenly equations constitute an integrable system. Thus the idea arose that $\mathcal{H}$-space might be a universal integrable system (Mason 1990). However, the reduction of the algebra of Hamiltonian vector fields over a symplectic manifold Σ^2, $\mathrm{sdiff}(\Sigma^2)$ to finite-dimensional algebras such as $su(N)$ does not exist for $N > 2$.

Consequently, it seems that Ward's conjecture should be extended to the algebras that include all finite-dimensional Lie algebras $sl(N, C)$ as well as the algebra of Hamiltonian vector fields. This is the point where *deformation quantization* enters into the theory of integrable systems.

The idea of deformation quantization, introduced by Bayen *et al* (1978) is to consider a deformed algebra of smooth functions on a classical phase space. The introduced associative $*$-product of two functions f, g is a formal power series in deformation parameter $\hbar$, $f * g = \sum \hbar^k \Delta_k(f, g), k \geqslant 0$. The $*$-product is assumed to satisfy the following axioms:

- It is local, i.e. $\Delta_k(f, g)$ depends only on f, g and partial derivatives of f, g are of rank not greater than k.
- It is a deformation of Poisson algebra, i.e. $\Delta_1(f, g) - \Delta_1(g, f) = \mathrm{i}\{f, g\}_{\mathrm{Poisson}}$.

One can prove that such a product exists on any symplectic (De Wilde and Lecomte 1983, Fedosov 1994) or even Poisson manifold (Kontsevich 1997).

It seems useful to consider integrable (quantum) deformations of integrable systems (Kupershmidt 1990, Strachan 1992, 1997, Takasaki 1994). It is so, because the Moyal bracket algebra can be reduced to all $su(N)$ algebras (Fairlie *et al* 1990). In a natural way the Poisson algebra is embedded in deformed algebra. This suggests that SDYM equations for $*$-bracket Lie algebra ($*$-SDYM equations) are reducible to $su(N)$-SDYM equations as well as to heavenly equations. This is the problem which we intend to consider in the present and the following papers. The present paper is devoted mainly to the formal problem of integrability of $*$-SDYM equations. In order to make our results more general we deal with an arbitrary $*$-product, and the Yang–Mills fields are defined on four-dimensional heavenly space.

It turns out that to have the formalism sufficiently general, one needs to deal with formal power series containing all negative powers of the deformation parameter $\hbar$, in particular with the power series of the form $\exp\left[\frac{1}{\mathrm{i}\hbar}A\right]$. As is well known (Fedosov 1996) such power series are well defined only for some special A. To ensure the existence of the exponent for a wide class of A we introduce a new formal parameter t (the *convergence parameter*). It is obvious that in applications only those series will be used that are convergent with respect to the parameter t.

Our paper is organized as follows. In section 1 we give some fundamental definitions and properties of formal power series. Then we obtain a group e^{Q} of formal power series suitable for the construction of respective gauge theory. The $*$-SDYM equations on Kähler manifold in the case of heavenly space are reduced to one equation called the *master equation* (ME) (1.16). In section 2 we find two collections of conserved charges (2.4) and (2.7). As is pointed out, the collection (2.7) is characteristic for any SDYM system and (2.4) is a generalization of hidden symmetries of heavenly or SDYM equations. We obtain two Lax pairs and the forward Penrose–Ward transform for ME. A dressing operator connecting those two pairs, and finally, the algebra of hidden symmetries are given. Section 3 is devoted to the solution of Riemann–Hilbert problem. We define the homogeneous Hilbert problem and show the existence of the solution of this problem for the formal power series group e^{Q} (*Birkhoff's factorization theorem*). Then the inverse Penrose–Ward transform is considered. Concluding remarks (section 4) close the paper.

Some applications of master equation (ME) in the theory of integrable systems and complex relativity will be presented in a forthcoming paper. In that paper, a sequence of $su(N)$ chiral fields tending to the heavenly space for $N \to \infty$ has been constructed. It has also been shown that any analytic solution of $su(N)$-SDYM equations can be obtained from some solution of $*$-SDYM equations.

1. Formal power series and $*$-SDYM equations

At the beginning of this section, we briefly summarize the basic definitions and theorems concerning formal power series. We define algebra, group and adjoint action. For more details, see MacLane (1939), Neumann (1949), Jacobson (1980) and Ruiz (1993).

The *ordered Abelian group* is a pair $((G, +), P)$, where $(G, +)$ is an Abelian group, P is a subset of G such that

- $0 \notin P$, $P \cap -P = \emptyset$ (0 is the neutral element of $(G, +)$);
- $\forall g, h \in P$, $g + h \in P$;
- $G = -P \cup \{0\} \cup P$.

We call P the subset of positive elements. It allows one to order elements of the group, i.e., if $g, h \in G$ we say that g is less than h and denote $g < h$ if and only if $h - g \in P$.

Let $((G, +), P)$ be an ordered group and $\boldsymbol{K}$ a vector space. Formal power series over G with coefficients in $\boldsymbol{K}$ is a map $a : G \to \boldsymbol{K}$, such that its support $\operatorname{supp} a = \{g \in G : a(g) \neq 0\}$ has the least element.

The formal power series a will be written in the following form:

$$a = \sum_{g\in G} a_g \hbar^g \qquad \text{where} \quad a_g = a(g), \quad \hbar\text{-parameter}.$$

The set of all formal power series over G with coefficients in $\boldsymbol{K}$ will be denoted by $\boldsymbol{K}((\hbar^G))$. It is a vector space over complex field, with addition and multiplication by scalar are defined pointwise by

$$a + b = \sum_{g\in G} (a_g + b_g)\hbar^g, \qquad \forall \alpha \in \boldsymbol{C} \quad \alpha a = \sum_{g\in G} \alpha a_g \hbar^g.$$

Moreover, if the pair $(\boldsymbol{K}, \circ)$ is an algebra then the multiplication of series is defined as

$$ab = \sum_{g\in G} \left(\sum_{h\in G} a_h \circ b_{g-h} \right) \hbar^g.$$

This multiplication is well defined, as the support of each series has the least element, so $\forall g \in G$ the number of elements $a_h \circ b_{g-h}$, $h \in G$ is finite.

In the case when $(G, +)$ is a group $(Z, +)$ we will write $\boldsymbol{K}((\hbar))$. Moreover, $\boldsymbol{K}[[\hbar]] := \{a \in \boldsymbol{K}((\hbar)), a_g = 0 \forall g \in -P\}$.

According to Fedosov's works (Fedosov 1994, 1996) the pair $(\mathcal{O}(\Sigma^{2n})[[\hbar]], *)$ constitutes an algebra. $\mathcal{O}(\Sigma^{2n})[[\hbar]]$ denotes linear space of formal power series with coefficients being holomorphic functions over symplectic manifold (Σ^{2n}, ω). $*$ is an associative and noncommutative multiplication $* : \mathcal{O}(\Sigma^{2n})[[\hbar]] \times \mathcal{O}(\Sigma^{2n})[[\hbar]] \to \mathcal{O}(\Sigma^{2n})[[\hbar]]$. The $*$-product considered is a *closed* $*$-product, i.e., the trace $\operatorname{Tr}(f * g) := \int \frac{\omega^n}{n!} f * g$ has the property $\operatorname{Tr}(f * g) = \operatorname{Tr}(g * f)$ (Connes *et al* 1992, Omori *et al* 1992, Fedosov 1996).

We can define the Lie algebra $(\mathcal{O}(\Sigma^{2n})[[\hbar]], \{,\})$ based on the $*$-product

$$\forall a, b \in \mathcal{O}(\Sigma^{2n})[[\hbar]] \qquad \{a, b\} = \frac{1}{\mathrm{i}\hbar}(a * b - b * a).$$

Our aim is to construct gauge theory. The fundamental object is the gauge group. The group element appears as an exponent of the element of Lie algebra. In the finite-dimensional case the exponent of left-invariant vector fields is the maximal integral curve, a 1-parameter subgroup of the Lie group.

In the case of the $*$-algebra taking an exponent is possible only for some special vectors (cf Fedosov (1996), Asakawa and Kishimoto (2000)). Such a group will not be general enough to define the gauge transformation.

In order to make superpositions of formal power series well defined, we need to introduce another parameter. So we would consider formal power series over $(Z, +)$ with coefficients in a space $\mathcal{O}(\Sigma^{2n})((\hbar))$. In a space $\mathcal{O}(\Sigma^{2n})((\hbar, t))$ of all such series we consider a subspace $\mathcal{A}$

$$\mathcal{A} = \left\{ A \in \mathcal{O}(\Sigma^{2n})((\hbar, t)), A = \sum_{m=0}^{\infty} \sum_{k=-m}^{\infty} t^m \hbar^k A_{m,k}(x) \right\}.$$

The star product $*$ defined on $\mathcal{O}(\Sigma^{2n})[[\hbar]]$ can be extended to $\mathcal{A}$ (we use the same symbol)

$$\forall A, B \in \mathcal{A}, \quad A = \sum_{m_1=0}^{\infty} \sum_{k_1=-m_1}^{\infty} t^{m_1} \hbar^{k_1} A_{m_1,k_1}(x), \quad B = \sum_{m_2=0}^{\infty} \sum_{k_2=-m_2}^{\infty} t^{m_2} \hbar^{k_2} B_{m_2,k_2}(x)$$

$$A * B = \sum_{m_1,m_2=0}^{\infty} t^{m_1+m_2} \hbar^{-(m_1+m_2)} \sum_{k_1=-m_1,k_2=-m_2}^{\infty} \left(\hbar^{k_1+m_1} A_{m_1,k_1}(x) * \hbar^{k_2+m_2} B_{m_2,k_2}(x) \right).$$

The algebra $(\mathcal{A}, *)$ is called a *formal* $*$-algebra.

Let $A \in \mathcal{A}$. The element $A(0)$, i.e. the one which stands at t^0, will be denoted as $\phi(A)$ and called the *free element*

$$\forall A \in \mathcal{A} \qquad A = \phi(A) + \sum_{m=1}^{\infty} \sum_{k=-m}^{\infty} t^m \hbar^k A_{m,k}, \qquad \text{where} \quad \phi(A) = \sum_{k=0}^{\infty} \hbar^k A_{0,k}.$$

The family $\mathcal{N}$ of formal power series belonging to a formal $*$-algebra $\mathcal{A}, \mathcal{N} = \{A_\delta \in \mathcal{A}, \delta \in \Omega\}$ is called *t-locally finite* if for each natural m the number of formal power series of this family having non-zero element at t^m is finite.

Then for each t-locally finite family and any family of complex number $\{a_\delta \in \boldsymbol{C}, \delta \in \Omega\}$ the sum $\sum_{\delta\in\Omega} a_\delta A_\delta$ is well defined:

- Let $A \in \mathcal{A}$ and let free element $\phi(A) = 0$. Then the family $\{A^n, n = 1, 2, \ldots\}$, where $A^n \equiv A * A^{n-1} = A^{n-1} * A$ is t-locally finite.
- Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a complex power series of one variable and $A \in \mathcal{A}$ with $\phi(A) = 0$. We define $f(A) = \sum_{n=0}^{\infty} a_n A^n$ (where A^n is as above).
- If $f_1(z) = \sum_{n=0}^{\infty} a_n z^n$ and $f_2(z) = \sum_{m=0}^{\infty} b_m z^m$ are two formal power series and $A \in \mathcal{A}$ with $\phi(A) = 0$, then

$$(f_1 \cdot f_2)(A) = (f_2 \cdot f_1)(A) = f_1(A) * f_2(A).$$

  This follows from the fact that $\mathcal{A}$ is an algebra with $A^n * A^m = A^m * A^n = A^{n+m}$, and the multiplication of coefficients a_n, b_m is commutative.
- For each $A \in \mathcal{A}$ such that $\phi(A) = 1$ there exists inversion of A, i.e. $X \in \mathcal{A}, A * X = X * A = 1$. Indeed, let $f_1(z) := (z+1)^{-1} = \sum_{n=0}^{\infty} (-1)^n z^n$. Let $f_2(z) := z + 1$, then $(f_1 \cdot f_2)(z) = 1$. If one defines $X = f_1(A - 1)$,

$$1 = (f_1 \cdot f_2)(A-1) = f_1(A-1) * f_2(A-1) = f_1(A-1) * A$$
$$1 = (f_2 \cdot f_1)(A-1) = f_2(A-1) * f_1(A-1) = A * f_1(A-1).$$

  In what follows we will write $A^{-1} := X$.

Remarks

- $\phi(A^{-1}) = 1$.
- The set $\{A \in \mathcal{A}, \phi(A) = 1\}$ with $*$-product form a group, it is a subgroup of invertible elements.

  Let $f_1(z) = \sum_{n=0}^{\infty} a_n z^n$, $f_2(z) = \sum_{m=1}^{\infty} b_m z^m$ and $A \in \mathcal{A}$ with $\phi(A) = 0$.
- Then superposition of series $f(A) = f_1(f_2(A))$ is well defined.

  This follows from the fact that the family $\{[f_2(A)]^n, n = 0, 1, 2, \ldots\}$ is t-locally finite.

Corollary 1.1. *For each $A \in \mathcal{A}$ with $\phi(A) = 0$, $\mathrm{e}^A := \sum_{n=0}^{\infty} \frac{A^n}{n!}$ is an element of the group with the free element equal to 1. On the other hand each element of this group is an exponent. The inversion map is given by $A = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(\mathrm{e}^A - 1)^n$.*

In what follows $\mathcal{Q}$ is subalgebra of formal power series with the free element equal to zero,

$$\mathcal{Q} := \left\{ A \in \mathcal{A}, A = \sum_{m=1}^{\infty} \sum_{k=-m}^{\infty} t^m \hbar^k A_{m,k}(x) \right\}.$$

$\mathrm{e}^{\mathcal{Q}}$ is a group of formal power series with the free element equal to 1,

$$\mathrm{e}^{\mathcal{Q}} := \left\{ a \in \mathcal{A}, a = 1 + \sum_{m=1}^{\infty} \sum_{k=-m}^{\infty} t^m \hbar^k a_{m,k}(x) \right\}.$$

From corollary 1.1 the algebra $\mathcal{Q}$ with Lie bracket $\forall A, B \in \mathcal{Q}$ $\{A, B\} := \frac{1}{\mathrm{i}\hbar}(A * B - B * A)$ will be called the Lie algebra of the group $\mathrm{e}^{\mathcal{Q}}$.

It is worth noting here that we do not consider differential structure on the group $\mathrm{e}^{\mathcal{Q}}$, so this is not a Lie group. Apart from that, for $A, B \in \mathcal{Q}$ one has

$$\frac{1}{\mathrm{i}\hbar} \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} (\mathrm{e}^{-\sqrt{\varepsilon}A} * \mathrm{e}^{-\sqrt{\varepsilon}B} * \mathrm{e}^{\sqrt{\varepsilon}A} * \mathrm{e}^{\sqrt{\varepsilon}B}) = \{A, B\}.$$

This justifies our notation.

From corollary 1.1 $\forall a \in \mathrm{e}^{\mathcal{Q}}$ there exists $\tilde{A} \in \mathcal{Q}$, such that $a = \exp(\tilde{A})$. For traditional reasons, we will write

$$a = \mathrm{e}^{\frac{1}{\mathrm{i}\hbar}A} \qquad \text{where} \quad \tilde{A} = \frac{1}{\mathrm{i}\hbar} \sum_{m=1}^{\infty} \sum_{k=-m+1}^{\infty} t^m \hbar^k A_{m,k}(X) =: \frac{1}{\mathrm{i}\hbar} A. \tag{1.1}$$

For our purpose, it is important to consider the following left actions of $\mathrm{e}^{\mathcal{Q}}$ on the algebra $\mathcal{A}$,

$$\psi : \mathrm{e}^{\mathcal{Q}} \times \mathcal{A} \to \mathcal{A}, \qquad \psi(a, f) := a * f \tag{1.2}$$

$$\phi : \mathrm{e}^{\mathcal{Q}} \times \mathcal{A} \to \mathcal{A}, \qquad \phi(a, f) := f * a^{-1} \tag{1.3}$$

and the adjoint representation

$$\mathrm{Ad} : \mathrm{e}^{\mathcal{Q}} \times \mathcal{A} \to \mathcal{A}, \qquad \mathrm{Ad}(a, f) := a * f * a^{-1}. \tag{1.4}$$

According to (1.1) the adjoint representation can be written in the following form:

$$a * f * a^{-1} = f + \sum_{l=1}^{\infty} \frac{1}{l!} \underbrace{\{A, \ldots, \{A}_{l\text{-times}}, f\} \cdots\}. \tag{1.5}$$

Each of the actions does not change the free element; see Asakawa and Kishimoto (2000).

1.1. Bundle of formal $*$-algebras

Let us consider four-dimensional complexified Kähler manifold $\mathcal{M}$.[3] We will consider functions and tensors on $\mathcal{M}$ which takes values in formal $*$-algebra $\mathcal{A}$.

Let $\mathcal{P}(\mathcal{M}, \mathrm{e}^{\mathcal{Q}})$ denote a trivial principle bundle $\pi_{\mathcal{P}} : \mathcal{P} \to \mathcal{M}$. In a total space $\mathcal{P}$ the structure group acts to the right in each fibre $\mathcal{P} \times \mathrm{e}^{\mathcal{Q}} \ni (u, c) \mapsto u * c \in \mathcal{P}$. The global sections are then

$$\sigma = 1 + \sum_{m=1}^{\infty} \sum_{k=-m}^{\infty} t^m \hbar^k \sigma_{m,k}(x, z^i) \quad \text{where} \quad \sigma_{m,k}(x, z^i) \in \mathcal{O}(\Sigma^{2n} \times \mathcal{M}).$$

Each representation of the group $\mathrm{e}^{\mathcal{Q}}$ in algebra $\mathcal{A}$ defined by (1.2), (1.3) and (1.4) allows one to define an associated bundle with typical fibre $\mathcal{A}$. In the Cartesian product $\mathcal{P} \times \mathcal{A}$ we introduce the following equivalence relations:

$$\begin{aligned}
(u, v) \sim_{\psi} (u', v') \quad &\Leftrightarrow \quad [\pi_{\mathcal{P}}(u) = \pi_{\mathcal{P}}(u'), \exists c \in \mathrm{e}^{\mathcal{Q}}, u' = u * c, v' = c^{-1} * v],\\
(u, v) \sim_{\phi} (u', v') \quad &\Leftrightarrow \quad [\pi_{\mathcal{P}}(u) = \pi_{\mathcal{P}}(u'), \exists c \in \mathrm{e}^{\mathcal{Q}}, u' = u * c, v' = v * c],\\
(u, v) \sim_{\mathrm{Ad}} (u', v') \quad &\Leftrightarrow \quad [\pi_{\mathcal{P}}(u) = \pi_{\mathcal{P}}(u'), \exists c \in \mathrm{e}^{\mathcal{Q}}, u' = u * c, v' = c^{-1} * v * c].
\end{aligned}$$

Definition 1.2

- *The formal $*$-algebra bundle $E(\mathcal{M}, \mathcal{A})$ over $\mathcal{M}$ is a set of equivalence classes of Cartesian product $\mathcal{P} \times \mathcal{A}$ in relation $\sim_{\psi}$, i.e. the total space is $E := \mathcal{P} \times \mathcal{A}/\sim_{\psi}$.*

Analogously,

- *the bundle $E' := \mathcal{P} \times \mathcal{A}/\sim_{\phi}$;*
- *the adjoint bundle $\mathrm{adj}(E) := \mathcal{P} \times \mathcal{A}/\sim_{\mathrm{Ad}}$.*

The map $\pi_{\mathcal{P}}$ induces the maps $\pi_E : E \to \mathcal{M}, \pi_{E'} : E' \to \mathcal{M}, \pi_{\mathrm{adj}(E)} : \mathrm{adj}(E) \to \mathcal{M}$. These maps send the equivalence class $[(u, v)]$ to $\pi_{\mathcal{P}}(u) \in \mathcal{M}$ which makes that $E, E', \mathrm{adj}(E)$ are bundles.

[3] The coordinates on $\mathcal{M}$ are denoted by $(w, z, \tilde{w}, \tilde{z})$. We also use the following abbreviations: $z^{\alpha} = \{w, z\}$, $z^{\tilde{\alpha}} = \{\tilde{w}, \tilde{z}\}$ and $z^i = \{w, z, \tilde{w}, \tilde{z}\}$. $\mathcal{M}$ is Hermitian manifold, i.e., is equipped with holomorphic nondegenerate metric $\mathrm{d}s^2 = 2g_{\alpha\tilde{\beta}}\,\mathrm{d}z^{\alpha} \otimes_s \mathrm{d}z^{\tilde{\beta}}$. This reduces the allowed transformations to those which preserve the foliation $w = \text{const}$, $z = \text{const}$ as well as $\tilde{w} = \text{const}$, $\tilde{z} = \text{const}$. Each 1-form $\sigma \in \Lambda^1\mathcal{M}$ can be decomposed to a sum $\sigma = \sigma_{(1,0)} + \sigma_{(0,1)}$ where $\sigma_{(1,0)} = \sigma_w\,\mathrm{d}w + \sigma_z\,\mathrm{d}z$, and $\sigma_{(0,1)} = \sigma_{\tilde{w}}\,\mathrm{d}\tilde{w} + \sigma_{\tilde{z}}\,\mathrm{d}\tilde{z}$. Analogously, the exterior derivative d is a sum of two Dolbeaut operators $\mathrm{d} = \partial + \tilde{\partial}$ where

$$\partial = \mathrm{d}w \wedge \partial_w + \mathrm{d}z \wedge \partial_z, \qquad \tilde{\partial} = \mathrm{d}\tilde{w} \wedge \partial_{\tilde{w}} + \mathrm{d}\tilde{z} \wedge \partial_{\tilde{z}}.$$

The *Kähler form* is a 2-form of the type $(1, 1)$, given by $\Omega = g_{\alpha\tilde{\beta}}\,\mathrm{d}z^{\alpha} \wedge \mathrm{d}\tilde{z}^{\beta}$. For *complexified Kähler manifold* the Kähler form is closed $\mathrm{d}\Omega = 0$. Locally, this means that $\Omega = \partial\tilde{\partial}\mathcal{K}$ for some complex function $\mathcal{K} = \mathcal{K}(w, z, \tilde{w}, \tilde{z})$ called *Kähler potential*. The Kähler form $\Omega = g_{\alpha\tilde{\beta}}\,\mathrm{d}z^{\alpha} \wedge \mathrm{d}z^{\tilde{\beta}}$ gives rise to the *volume element* $\boldsymbol{\nu} := \frac{1}{2}\Omega \wedge \Omega = g\,\mathrm{d}w \wedge \mathrm{d}\tilde{w} \wedge \mathrm{d}z \wedge \mathrm{d}\tilde{z}$ where $g = \det(g_{\alpha\tilde{\beta}}) = \mathcal{K},_{w\tilde{w}}\,\mathcal{K},_{z\tilde{z}} - \mathcal{K},_{w\tilde{z}}\,\mathcal{K},_{z\tilde{w}}$. Under the Hodge duality the following 2-forms constitute the basis of anti-self-dual forms

$$\Sigma^{\dot{1}\dot{1}} := \mathrm{d}\tilde{x} \wedge \mathrm{d}\tilde{y}, \qquad \Sigma^{\dot{1}\dot{2}} := \Omega, \qquad \Sigma^{\dot{2}\dot{2}} := \mathrm{d}x \wedge \mathrm{d}y$$

(see Plebański (1975), Flaherty (1976), Ko *et al* (1981), Mason and Woodhouse (1996) for more details.)

For each section $\sigma : \mathcal{M} \to \mathcal{P}$ of the principal bundle, the section $f : \mathcal{M} \to E$ of the associated bundle induces map $\tilde{f}_\sigma : \mathcal{P} \to \mathcal{P} \times \mathcal{A}$ such that the following diagram is commutative,

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{\tilde{f}_\sigma} & \mathcal{P} \times \mathcal{A} \\ \sigma \big\uparrow & & \big\downarrow \ell \\ \mathcal{M} & \xrightarrow{f} & \mathrm{E} \end{array}$$

where ℓ is the canonical map $\ell(u, v) = [(u, v)]$.

The map $\tilde{f}_\sigma$ defines another map $f_\sigma : \mathcal{M} \to \mathcal{A}$ by $f_\sigma := pr_{\mathcal{A}} \circ \tilde{f}_\sigma \circ \sigma$. It is called the representation of the section $f : \mathcal{M} \to E$ with respect to the section $\sigma : \mathcal{M} \to \mathcal{P}$.

Higher external powers of the bundle E, $E^k := E \otimes \Lambda^k \mathcal{M}\ k = 1, 2, 3, 4$ form an algebra of the forms with values in E

$$E \otimes \Lambda\mathcal{M} := \bigoplus_{k=1}^{4} E \otimes \Lambda^k \mathcal{M},$$

with external product (in terms of representations)

$$\boldsymbol{\omega} \wedge \boldsymbol{\nu} := \omega_{i_1 \ldots i_k} * \nu_{j_1 \ldots j_l}\, \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k} \wedge \mathrm{d}x^{j_1} \wedge \cdots \wedge \mathrm{d}x^{j_l}.$$

This allows us to define the bracket of forms

$$\begin{aligned} \{\boldsymbol{\omega}, \boldsymbol{\nu}\} &:= \frac{1}{\mathrm{i}\hbar}(\boldsymbol{\omega} \wedge \boldsymbol{\nu} - (-1)^{kl} \boldsymbol{\nu} \wedge \boldsymbol{\omega}) \\ &= \{\omega_{i_1 \ldots i_k}, \nu_{j_1 \ldots j_l}\}\, \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k} \wedge \mathrm{d}x^{j_1} \wedge \cdots \wedge \mathrm{d}x^{j_l}. \end{aligned}$$

Definition 1.3

- *A connection in E is a linear map* $D : \mathrm{Sec}(E) \to \mathrm{Sec}(E^1)$, *defined by*

$$\forall f \in \mathrm{Sec}(E) \qquad Df_\sigma := \mathrm{d}f_\sigma + \frac{1}{\mathrm{i}\hbar} A_\sigma * f_\sigma \tag{1.6}$$

  where A_σ *is a local 1-form with values in* $\mathcal{Q}$, *the Lie algebra of the group* $\mathrm{e}^{\mathcal{Q}}$, *and* $f_\sigma : \mathcal{M} \to \mathcal{A}$ *is a representation of the section* f.
- *Any connection D in the bundle E induces connections in bundles* E' *and* $\mathrm{adj}(E)$, *respectively (we will use the same symbol D as in each case the connection is defined by the same 1-form* A_σ*)*

$$\begin{aligned} &\forall f \in \mathrm{Sec}(E') &\qquad Df_\sigma &:= \mathrm{d}f_\sigma + f_\sigma * \frac{1}{\mathrm{i}\hbar} A_\sigma \\ &\forall f \in \mathrm{Sec}(\mathrm{adj}(E)) &\qquad Df_\sigma &:= \mathrm{d}f_\sigma + \{A_\sigma, f_\sigma\}. \end{aligned}$$

If f_σ, f_ρ are two representations of the same section $f : \mathcal{M} \to E$ then there exists $c : \mathcal{M} \to \mathrm{e}^{\mathcal{Q}}$, such that $f_\rho = c^{-1} * f_\sigma$. Thus for $Df_\rho = \mathrm{d}f_\rho + \frac{1}{\mathrm{i}\hbar} A_\rho * f_\rho$ one gets

$$Df_\rho = c^{-1} * \left[\mathrm{d}f_\sigma + \frac{1}{\mathrm{i}\hbar} A_\sigma * f_\sigma\right] \qquad \text{where} \quad A_\sigma = c * A_\rho * c^{-1} + \mathrm{i}\hbar c * \mathrm{d}c^{-1}. \tag{1.7}$$

The transformation law (1.7) can be rewritten for $c^{-1} = \mathrm{e}^{\frac{1}{\mathrm{i}\hbar}B}$

$$A_\rho = A_\sigma + \sum_{n=1}^{\infty} \frac{1}{n!} \underbrace{\{B, \ldots, \{B,}_{n\ \text{times}} A_\sigma\} \cdots\} - \mathrm{d}B - \sum_{n=1}^{\infty} \frac{1}{(n+1)!} \underbrace{\{B, \cdots \{B,}_{n\ \text{times}} \mathrm{d}B\} \cdots\}.$$

In what follows we will omit the superscripts σ, ρ denoting sections of the principal bundle. For different representations we will use more common symbols $'$, $''$, etc.

The connection in E may be extended, in a natural way, to E^k. We denote the *exterior covariant differentiation* by the same symbol $D : \mathrm{Sec}(E^k) \to \mathrm{Sec}(E^{k+1})$ defined for any $\omega \in \mathrm{Sec}(E^k)$ by $D\omega = \mathrm{d}\omega + \frac{1}{\mathrm{i}\hbar} A \wedge \omega$.

The *curvature* of D is a map $D \circ D \equiv D^2 : \mathrm{Sec}(E) \to \mathrm{Sec}(E^2)$. The form $F := DA$ is a *curvature form* (where operator $D := d + \frac{1}{\mathrm{i}\hbar} A\wedge$)

$$F := \mathrm{d}A + \frac{1}{\mathrm{i}\hbar} A \wedge A = \mathrm{d}A + \frac{1}{2}\{A, A\}.$$

For each $\omega \in \mathrm{Sec}(E^k)$ we have $D^2\omega = F \wedge \omega$.

1.2. Self-dual Yang–Mills equations

The following 2-forms constitute the basis of anti-self-dual forms

$$\Sigma^{\dot{1}\dot{1}} := \mathrm{d}\tilde{w} \wedge \mathrm{d}\tilde{z}, \qquad \Sigma^{\dot{1}\dot{2}} := \Omega, \qquad \Sigma^{\dot{2}\dot{2}} := \mathrm{d}w \wedge \mathrm{d}z$$

where $\Omega = g_{\alpha\tilde{\beta}}\, \mathrm{d}z^\alpha \wedge \mathrm{d}z^{\tilde{\beta}}$ is the Kähler form. Then the curvature form F is self-dual iff

$$F \wedge \Sigma^{\dot{1}\dot{1}} = 0, \quad F \wedge \Sigma^{\dot{1}\dot{2}} = 0, \quad F \wedge \Sigma^{\dot{2}\dot{2}} = 0.$$

The self-dual Yang–Mills (SDYM) equations read

$$\partial_w A_z - \partial_z A_w + \{A_w, A_z\} = 0, \tag{1.8}$$

$$\partial_{\tilde{w}} A_{\tilde{z}} - \partial_{\tilde{z}} A_{\tilde{w}} + \{A_{\tilde{w}}, A_{\tilde{z}}\} = 0, \tag{1.9}$$

$$g^{\tilde{\beta}\alpha}(\partial_\alpha A_{\tilde{\beta}} - \partial_{\tilde{\beta}} A_\alpha + \{A_\alpha, A_{\tilde{\beta}}\}) = 0. \tag{1.10}$$

The first two of the above equations can be interpreted as integrability conditions for substitutions $A_\alpha = \mathrm{i}\hbar a^{-1} * \partial_\alpha a$, $A_{\tilde{\alpha}} = \mathrm{i}\hbar b^{-1} * \partial_{\tilde{\alpha}} b$ where $a, b : \mathcal{M} \to \mathrm{e}^{\mathcal{Q}}$. Then the third equation takes the form

$$\mathrm{i}\hbar a^{-1} * g^{\tilde{\beta}\alpha} \partial_\alpha [a * b^{-1} * \partial_{\tilde{\beta}}(b * a^{-1})] * a = 0.$$

After the substitution $J := b * a^{-1}$ this becomes *Yang's equation* (Yang 1977, Parkes 1992)

$$g^{\tilde{\beta}\alpha} \partial_\alpha (J^{-1} * \partial_{\tilde{\beta}} J) = 0, \tag{1.11}$$

or equivalently $\partial\tilde{\partial}\mathcal{K} \wedge \partial(J^{-1} * \tilde{\partial} J) = 0$ where $\partial = \mathrm{d}z^\alpha \wedge \partial_\alpha$, $\tilde{\partial} = \mathrm{d}z^{\tilde{\alpha}} \wedge \partial_{\tilde{\alpha}}$ are Dolbeault operators and $\mathcal{K}$ is the Kähler potential, i.e. $\Omega = \partial\tilde{\partial}\mathcal{K}$. This equation arises from a minimum action principle for $S = \frac{1}{2}\kappa \int \omega^n \mathcal{K} F \wedge F$ where F is a curvature form of the connection $A = \mathrm{i}\hbar J^{-1} * \tilde{\partial} J$, and ω is symplectic form on Σ^{2n} (Donaldson 1985, Nair and Schiff 1990, Mason and Woodhouse 1996).

In our consideration, we will work in the so-called K-formalism of Newman (Newman 1978, Leznov 1988, Parkes 1992, Plebański and Przanowski 1996, Mason and Woodhouse 1996).

Choosing gauge such that $A = \mathrm{i}\hbar J^{-1} * \tilde{\partial} J$, the SDYM equations reduce to Yang's equation (1.11). It can be rewritten in the form

$$g^{\tilde{\beta}\alpha} \nabla_\alpha A_{\tilde{\beta}} = 0 \quad \Leftrightarrow \quad \partial_\beta (g g^{\tilde{\alpha}\beta} A_{\tilde{\alpha}}) = 0 \tag{1.12}$$

where ∇_α is the covariant derivative with respect to the Levi-Civita connection[4] on $\mathcal{M}$. Equation (1.12) is equivalent to the existence of Ξ such that

$$\partial_\beta \Xi = \epsilon_{\beta\gamma} g g^{\tilde{\alpha}\gamma} A_{\tilde{\alpha}} \quad \Rightarrow \quad A_{\tilde{\alpha}} = \frac{1}{g}\epsilon^{\beta\gamma} g_{\gamma\tilde{\alpha}} \partial_\beta \Xi = -\epsilon_{\tilde{\alpha}\tilde{\beta}} g^{\tilde{\beta}\gamma} \partial_\gamma \Xi \tag{1.13}$$

where $\epsilon^{\alpha\beta}$ is a tensor density in (w, z) variables, defined in each coordinate neighbourhood by $(\epsilon_{\alpha\beta}) := \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) =: (\epsilon^{\alpha\beta})$. Analogously $\epsilon_{\tilde{\alpha}\tilde{\beta}}$ is a tensor density in $(\tilde{w}, \tilde{z})$ variables $(\epsilon_{\tilde{\alpha}\tilde{\beta}}) := \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) =: (\epsilon^{\tilde{\alpha}\tilde{\beta}})$. The definition (1.13) of Ξ implies that under the change of variables $\tilde{w}' = \tilde{w}'(\tilde{w}, \tilde{z})$, $\tilde{z}' = \tilde{z}'(\tilde{w}, \tilde{z})$ Ξ transforms according to $\Xi' = \frac{\partial(\tilde{w}, \tilde{z})}{\partial(\tilde{w}', \tilde{z}')}\Xi$, i.e., Ξ is a scalar density in these variables. In this case covariant derivative $\nabla_{\tilde{\alpha}}$ acts on densities according to the rule $\nabla_{\tilde{\alpha}} \Xi = \partial_{\tilde{\alpha}} \Xi - (\ln g),_{\tilde{\alpha}} \Xi$ while $\nabla_\alpha \Xi = \partial_\alpha \Xi$.

Inserting $A_{\tilde{\alpha}}$ given by (1.13) into (1.9) one gets

$$g^{\tilde{\alpha}\beta} \nabla_{\tilde{\alpha}} \nabla_\beta \Xi + \frac{1}{2g} \epsilon^{\alpha\beta} \{\nabla_\alpha \Xi, \nabla_\beta \Xi\} = 0 \tag{1.14}$$

For the first time this equation was proposed by Park (1992) for Poisson algebra. In the case of Moyal algebra and flat base manifold it was considered by Plebański and Przanowski (1996), Przanowski and Formański (1999).

The linearized equation (1.14) reads

$$g^{\tilde{\alpha}\beta} \nabla_{\tilde{\alpha}} \nabla_\beta \eta + \frac{1}{g} \epsilon^{\alpha\beta} \{\nabla_\alpha \Xi, \nabla_\beta \eta\} = 0. \tag{1.15}$$

Let us consider a special case of the base manifold $\mathcal{M}$, i.e. the heavenly space. For a Kählerian manifold to be heavenly space it is necessary and sufficient that the Ricci tensor $R_{\alpha\tilde{\beta}}$ vanish (cf footnote 4), i.e. $(\ln g),_{\alpha\tilde{\beta}} = 0$. Thus the determinant $g = G(w, z)\tilde{G}(\tilde{w}, \tilde{z})$, and rewriting (1.13) once again

$$A_{\tilde{\alpha}} = \frac{1}{G} \epsilon^{\beta\gamma} g_{\gamma\tilde{\alpha}} \partial_\beta \left(\frac{\Xi}{\tilde{G}}\right) = -\tilde{G} \epsilon_{\tilde{\alpha}\tilde{\beta}} g^{\tilde{\beta}\gamma} \partial_\gamma \left(\frac{\Xi}{\tilde{G}}\right).$$

We can define $\Theta := \frac{\Xi}{\tilde{G}}$ which is now a scalar function. With the help of this function the master equation (1.14) now reads

$$g^{\tilde{\beta}\alpha} \partial_{\tilde{\beta}} \partial_\alpha \Theta + \frac{1}{2G} \epsilon^{\alpha\beta} \{\partial_\alpha \Theta, \partial_\beta \Theta\} = 0. \tag{1.16}$$

[4] On a Kählerian manifold the Hermitian connection is at the same time the Levi-Civita connection

$$\Gamma^\alpha_{\beta\gamma} = g^{\alpha\tilde{\sigma}} \partial_\gamma g_{\beta\tilde{\sigma}}, \qquad \Gamma^{\tilde{\alpha}}_{\tilde{\beta}\tilde{\gamma}} = g^{\tilde{\alpha}\sigma} \partial_{\tilde{\gamma}} g_{\sigma\tilde{\beta}}.$$

This implies that the only non-zero coefficients of the curvature tensor (apart from those obtained from symmetry operations) are

$$R_{\tilde{\alpha}\beta\gamma\tilde{\delta}} = -g_{\sigma\tilde{\alpha}} \partial_{\tilde{\delta}} \Gamma^\sigma_{\beta\gamma} = -g_{\beta\tilde{\sigma}} \partial_\gamma \Gamma^{\tilde{\sigma}}_{\tilde{\alpha}\tilde{\delta}}.$$

The Ricci tensor $R_{\alpha\tilde{\beta}} = g^{\tilde{\sigma}\gamma} R_{\gamma\tilde{\beta}\alpha\tilde{\sigma}} = g^{\tilde{\sigma}\gamma} R_{\alpha\tilde{\beta}\gamma\tilde{\sigma}} = (\ln g),_{\alpha\tilde{\beta}}$ and Ricci scalar $R = g^{\tilde{\beta}\alpha} R_{\alpha\tilde{\beta}}$.

The Weyl tensor of conformal curvature

$$C_{\tilde{\alpha}\beta\gamma\tilde{\delta}} = R_{\tilde{\alpha}\beta\gamma\tilde{\delta}} + \tfrac{1}{2}(g_{\beta\tilde{\delta}} R_{\gamma\tilde{\alpha}} + g_{\gamma\tilde{\alpha}} R_{\beta\tilde{\delta}}) - \tfrac{1}{6} R g_{\beta\tilde{\delta}} g_{\gamma\tilde{\alpha}}.$$

The manifold $(\mathcal{M}, \mathrm{d}s^2)$ is called *weak heaven* (Plebański 1975) or *right conformally flat* (Ko *et al* 1981) if the 2-forms $C_{\alpha\tilde{\beta}} := \frac{1}{2} C_{\alpha\tilde{\beta}\gamma\tilde{\delta}}\, \mathrm{d}z^\gamma \wedge \mathrm{d}z^{\tilde{\delta}}$ are self-dual. For these to be true the following conditions should be satisfied: $C_{\alpha\tilde{\beta}} \wedge \Sigma^{\dot{A}\dot{B}} = 0$, $\dot{A}, \dot{B} = \dot{1}, \dot{2}$. But the only non-trivial condition is $C_{\alpha\tilde{\beta}} \wedge \Sigma^{\dot{1}\dot{2}} = \frac{1}{12} R g_{\alpha\tilde{\beta}} \boldsymbol{\nu}$ so R must vanish. The space $(\mathcal{M}, \mathrm{d}s^2)$ is *heavenly* if the curvature 2-forms are self-dual. As $\frac{1}{2} R_{\alpha\tilde{\beta}\gamma\tilde{\delta}}\, \mathrm{d}z^\gamma \wedge \mathrm{d}z^{\tilde{\delta}} \wedge \Sigma^{\dot{1}\dot{2}} = \frac{1}{2} R_{\alpha\tilde{\beta}} \boldsymbol{\nu}$ the Ricci tensor must vanish.

It will be called the *master equation* (ME). This equation arises from a minimum action principle for the action

$$S = \frac{1}{2}\int \omega^n \left[\tilde{G}\tilde{\epsilon} \wedge \frac{1}{3}\Theta * \{\partial\Theta, \partial\Theta\} + \Omega \wedge \partial\Theta \wedge \tilde{\partial}\Theta\right] \tag{1.17}$$

where $\tilde{\epsilon} := \frac{1}{2}\epsilon_{\tilde{\alpha}\tilde{\beta}}\, \mathrm{d}x^{\tilde{\alpha}} \wedge \mathrm{d}x^{\tilde{\beta}}$ and $\Omega = g_{\alpha\tilde{\beta}}\, \mathrm{d}z^{\alpha} \wedge \mathrm{d}z^{\tilde{\beta}}$ is the Kähler form. (Note that our notation is covariant and we do not work in any special coordinates like, for example, Plebański's coordinates (Plebański 1975) in which $g = 1$ (Parkes 1992).) The action (1.17) generalizes actions given by Boyer and Plebański (1985), Leznov (1988), Parkes (1992), Plebański and Przanowski (1996).

2. Conservation laws and twistor construction

In this section, we construct two hierarchies of conservation laws for ME on heavenly background (1.16). It is known from the previous section that in the algebra $\mathcal{A}$ there exist both the $*$-product and the bracket $\{\cdot, \cdot\}$. Hence one can expect the existence of two different linear systems for ME.

2.1. Hierarchy of hidden symmetries of ME

Let $\mathcal{W}$ denote the space of solutions to the master equation (ME) (1.16), and $T\mathcal{W}$ the space of solutions to the *linearized master equation* (LME)

$$g^{\tilde{\beta}\alpha}\partial_{\tilde{\beta}}\partial_{\alpha}\phi + \frac{1}{G}\epsilon^{\alpha\beta}\{\partial_{\alpha}\Theta, \partial_{\beta}\phi\} = 0. \tag{2.1}$$

Define two operators acting on functions on $\mathcal{M}$ with values in $\mathcal{A}$

$$\mathcal{L}^{\alpha} := g^{\tilde{\beta}\alpha}\partial_{\tilde{\beta}} + \frac{\epsilon^{\beta\alpha}}{G}\{\partial_{\beta}\Theta, \cdot\}, \qquad \alpha = 1, 2. \tag{2.2}$$

Then their commutator can be easily found to be

$$[\mathcal{L}^{w}, \mathcal{L}^{z}](\cdot) = \frac{1}{G}\left\{g^{\tilde{\beta}\alpha}\partial_{\tilde{\beta}}\partial_{\alpha}\Theta + \frac{1}{2G}\epsilon^{\alpha\beta}\{\partial_{\alpha}\Theta, \partial_{\beta}\Theta\}, \cdot\right\}.$$

So the operators $\mathcal{L}^{\alpha}$ commute iff Θ satisfies ME (1.16). Equation (2.1) written in terms of these operators reads

$$\mathcal{L}^{\alpha}\partial_{\alpha}\phi = 0.$$

Suppose that $\phi_{(0)} \in T\mathcal{W}$. Then define the *current* $J_{(1)}$ with components $J^{\alpha}_{(1)} := \mathcal{L}^{\alpha}\phi_{(0)}$, $J^{\tilde{\alpha}}_{(1)} = 0$. Thus $\nabla_i J^i_{(1)} = \nabla_{\alpha} J^{\alpha}_{(1)} = \nabla_{\alpha}\mathcal{L}^{\alpha}\phi_{(0)} = \mathcal{L}^{\alpha}\partial_{\alpha}\phi_{(0)}$. As $\phi_{(0)}$ solves the LME the current $J_{(1)}$ fulfils the conservation law $\nabla_{\alpha} J^{\alpha}_{(1)} = 0$. This conservation law can be written in the form $\partial_{\alpha}\big(G J^{\alpha}_{(1)}\big) = 0$, which implies the existence of the scalar function $\phi_{(1)}$, such that

$$G J^{\alpha}_{(1)} = -\epsilon^{\alpha\beta}\partial_{\beta}\phi_{(1)} \;\;\Rightarrow\;\; \partial_{\alpha}\phi_{(1)} = G\epsilon_{\alpha\beta} J^{\beta}_{(1)}. \tag{2.3}$$

This function gives rise to the next current $J^{\alpha}_{(2)} := \mathcal{L}^{\alpha}\phi_{(1)}$, divergence of which also vanishes, i.e.

$$\begin{aligned}\nabla_{\alpha} J^{\alpha}_{(2)} &= \nabla_{\alpha}\mathcal{L}^{\alpha}\phi_{(1)} = \mathcal{L}^{\alpha}\partial_{\alpha}\phi_{(1)} = \epsilon_{\alpha\beta}\mathcal{L}^{\alpha} G\mathcal{L}^{\beta}\phi_{(0)} \\ &= \left\{g^{\tilde{\beta}\alpha}\partial_{\tilde{\beta}}\partial_{\alpha}\Theta + \frac{1}{2G}\epsilon^{\alpha\beta}\{\partial_{\alpha}\Theta, \partial_{\beta}\Theta\}, \phi_{(0)}\right\} \overset{\text{by ME}}{=} 0.\end{aligned}$$

Note that the above equality also states that $\phi_{(1)} \in T\mathcal{W}$. One can repeat above construction starting from $\phi_{(1)}$. We are led to an iterative procedure. Given the nth conserved charge $\phi_{(n)}$

one constructs the (n+1) current $J^{\alpha}_{(n+1)} := \mathcal{L}^{\alpha}\phi_{(n)}$ and then one solves $\partial_\alpha\phi_{(n+1)} = G\epsilon_{\alpha\beta}J^{\beta}_{(n+1)}$ for $\phi_{(n+1)}$, i.e., $\phi_{(n+1)}$ is a solution of

$$\partial_\alpha\phi_{(n+1)} = G\epsilon_{\alpha\beta}\mathcal{L}^{\beta}\phi_{(n)}. \tag{2.4}$$

Such a solution is an element of $T\mathcal{W}$ and it defines a divergence-free current.

Remarks

- In this way, we define an integro-differential *recursion* operator $\phi_{(n+1)} = \mathcal{R}\phi_{(n)}$ depending additionally on boundary conditions imposed. This operator is invertible.
- The elements of $T\mathcal{W}$ are hidden symmetries of ME. These symmetries generalize the symmetries for heavenly equations obtained by Boyer and Plebański (1985), Strachan (1993), Husain (1994), Dunajski and Mason (2000).

2.2. *Second collection of conserved charges*

The existence of the $*$-product in each fibre of the bundle E allows us to construct another set of operators

$$\mathcal{D}^{\alpha} := g^{\tilde\beta\alpha}\partial_{\tilde\beta} + \frac{1}{\mathrm{i}\hbar}\frac{\epsilon^{\beta\alpha}}{G}\partial_\beta\Theta *$$

where Θ is a solution of ME (1.16). For any function with value in $\mathcal{A}$ we have $\nabla_\alpha\mathcal{D}^{\alpha} - \mathcal{D}^{\alpha}\partial_\alpha = 0$. Let $\eta_{(0)}$ be a solution to

$$\mathcal{D}^{\alpha}\partial_\alpha\eta_{(0)} = 0. \tag{2.5}$$

Then define vector $j^{\alpha}_{(1)} := \mathcal{D}^{\alpha}\eta_{(0)}$, $j^{\tilde\alpha} := 0$. The divergence of $j^{i}_{(1)}$ vanishes,

$$\nabla_i j^{i}_{(1)} = \nabla_\alpha j^{\alpha}_{(1)} = \nabla_\alpha\mathcal{D}^{\alpha}\eta_{(0)} = \mathcal{D}^{\alpha}\partial_\alpha\eta_{(0)} = 0.$$

In the same way as in the previous case, the conserved current $j^{i}_{(1)}$ defines a function $\eta_{(1)}$ by a system of equations

$$\partial_\alpha\eta_{(1)} = G\epsilon_{\alpha\beta}j^{\beta}_{(1)}. \tag{2.6}$$

This function fulfils

$$\begin{aligned}\mathcal{D}^{\alpha}\partial_\alpha\eta_{(1)} &= \mathcal{D}^{\alpha}G\epsilon_{\alpha\beta}j^{\beta}_{(1)} = \epsilon_{\alpha\beta}\mathcal{D}^{\alpha}G\mathcal{D}^{\beta}\eta_{(0)}\\ &= \frac{1}{\mathrm{i}\hbar}\left(g^{\tilde\beta\alpha}\partial_{\tilde\beta}\partial_\alpha\Theta + \frac{\epsilon^{\alpha\beta}}{2G}\{\partial_\alpha\Theta, \partial_\beta\Theta\}\right) * \eta_{(0)} \overset{\text{by ME}}{=} 0.\end{aligned}$$

The function $\eta_{(1)}$ fulfils the same equation as $\eta_{(0)}$. This allows us to define another current $j^{i}_{(2)}$, with components $j^{\alpha}_{(2)} = \mathcal{D}^{\alpha}\eta_{(1)}$, $j^{\tilde\alpha}_{(2)} = 0$. The divergence of $j_{(2)}$ vanishes.

Continuing this procedure we arrive at the series of conserved charges $\eta_{(0)}, \eta_{(1)}, \ldots$ and currents $j^{\alpha}_{(1)}, j^{\alpha}_{(2)}, \ldots$ defined by the recursion equations

$$\partial_\alpha\eta_{(n+1)} = G\epsilon_{\alpha\beta}j^{\beta}_{(n+1)} = G\epsilon_{\alpha\beta}\mathcal{D}^{\beta}\eta_{(n)}, \qquad n = 0, 1, \ldots. \tag{2.7}$$

Remarks

- As in the case of hidden symmetries the system (2.7) defines the recursion operator $\eta_{(n+1)} = \widetilde{\mathcal{R}}\eta_{(n)}$. The above hierarchy of conservation laws is characteristic for self-dual Yang–Mills equations (cf Brezin *et al* (1979), Prasad *et al* (1979), Chau (1983)).
- Both hierarchies were presented in Przanowski *et al* (2001a, 2001b) in the case of the complexified Minkowski space $\mathcal{M}$ and the Moyal $*$-product.

The characteristic feature of integrable systems besides the existence of infinite number of conservation laws is the existence of a Lax pair and some geometric construction related to the system considered. We are going to deal with this problem.

2.3. Twistors for $\mathcal{M}$

Twistor surface or *β-plane* or *null string* (Penrose 1976, Plebański and Hacyan 1975, Flaherty 1976, Ward and Wells 1990, Mason and Woodhouse 1996) is a two-dimensional submanifold $\mathcal{S} \subset \mathcal{M}$ such that

- $\mathcal{S}$ is totally null, i.e. $\forall p \in \mathcal{S}$ and $\forall v \in T_p\mathcal{S}\ \mathrm{d}s^2(v, v) = 0$;
- the 2-form orthogonal to $\mathcal{S}$ is anti-self-dual.

This implies that it is also totally geodesic, i.e. $\forall p \in \mathcal{S}$ and $\forall v \in T_p\mathcal{S}$ geodesic with tangent vector v in p lies on the surface $\mathcal{S}$.

For heavenly space $(\mathcal{M}, \mathrm{d}s^2)$ we have $\partial_\alpha \partial_{\tilde{\beta}} \ln g = 0$, i.e. $g = G(w, z)\tilde{G}(\tilde{w}, \tilde{z})$. In appropriate coordinates $g = 1$ and this is the *first heavenly equation* (Plebański 1975). We work in an arbitrary coordinate system, which means that the determinant g is a product of two functions.

For each $\lambda \in \boldsymbol{CP}^1 - \{\infty\}$ the integral 2-surface of two vector fields

$$\ell_w = \frac{\partial}{\partial w} - \lambda G g^{\tilde{\sigma} z} \frac{\partial}{\partial z^{\tilde{\sigma}}}, \qquad \ell_z = \frac{\partial}{\partial z} + \lambda G g^{\tilde{\sigma} w} \frac{\partial}{\partial z^{\tilde{\sigma}}} \tag{2.8}$$

is a twistor surface. This follows from the Frobenius theorem as those fields commute, and from the fact that $\mathrm{d}s^2(\ell_w, \ell_w) = \mathrm{d}s^2(\ell_w, \ell_z) = \mathrm{d}s^2(\ell_z, \ell_z) = 0$. The anti-self-dual form $\Sigma(\lambda) := \tilde{G}\,\mathrm{d}\tilde{w} \wedge \mathrm{d}\tilde{z} - \lambda\Omega + \lambda^2 G\,\mathrm{d}w \wedge \mathrm{d}z$ is orthogonal to the distribution $\mathcal{W}_\lambda = \mathrm{span}\{\ell_w, \ell_z\}$. Moreover, it is closed $\mathrm{d}\Sigma(\lambda) = 0$ and degenerate $\Sigma(\lambda) \wedge \Sigma(\lambda) = 0$. From Darboux theorems this allows one to introduce smooth functions P^w, P^z such that $\Sigma(\lambda)$ takes the canonical form $\Sigma(\lambda) = \mathrm{d}P^w \wedge \mathrm{d}P^z$.

Analogously in the domain $\boldsymbol{CP}^1 - \{0\} \ni \zeta$, the twistor surface is defined by

$$\underline{\ell_w} = \frac{1}{\zeta}\frac{\partial}{\partial w} - G g^{\tilde{\sigma} z} \frac{\partial}{\partial z^{\tilde{\sigma}}} \qquad \underline{\ell_z} = \frac{1}{\zeta}\frac{\partial}{\partial z} + G g^{\tilde{\sigma} w} \frac{\partial}{\partial z^{\tilde{\sigma}}} \tag{2.9}$$

and $\Sigma(\zeta) = \frac{1}{\zeta^2}\tilde{G}\,\mathrm{d}\tilde{x} \wedge \mathrm{d}\tilde{y} - \frac{1}{\zeta}\Omega + G\,\mathrm{d}x \wedge \mathrm{d}y$. The canonical form of $\Sigma(\zeta)$ reads $\Sigma(\zeta) = \mathrm{d}\underline{P^w} \wedge \mathrm{d}\underline{P^z}$.

On the patching $\boldsymbol{CP}^1 - \{0, \infty\}$ for $\lambda = \zeta$ the distributions considered are equivalent. Then we see that for each point $p \in \mathcal{M}$ and for each $\lambda \in \boldsymbol{CP}^1$ there exists a twistor surface through p (Penrose 1976). The set $\mathcal{PT}$ of all twistor surfaces is a three-dimensional complex manifold called the *projective twistor space*. It is covered by two coordinate neighbourhoods, $(V, (P^w, P^z, \lambda))$ for $\lambda \in \boldsymbol{CP}^1 - \{\infty\}$ and $(\underline{V}, (\underline{P^w}, \underline{P^z}, \zeta))$ for $\zeta \in \boldsymbol{CP}^1 - \{0\}$.

A more general result also holds, i.e., the projective twistor space exists iff $(\mathcal{M}, \mathrm{d}s^2)$ is a weak heaven (Penrose and Ward 1980).

Both manifolds $\mathcal{M}$ and $\mathcal{PT}$ are embedded in the so-called *correspondence space*, $\mathcal{F} := \mathcal{M} \times \boldsymbol{CP}^1$

$$\begin{tikzcd} & \mathcal{M} \times \boldsymbol{CP}^1 \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \mathcal{M} & & \mathcal{PT} \end{tikzcd}$$

2.4. The Lax pair and Penrose–Ward transform

In this section, we construct the formal bundle over twistor space $\mathcal{PT}$ which is determined by a solution Θ of master equation (ME). First we start with a Lax pair for ME. For each value of a spectral parameter belonging to $\boldsymbol{CP}^1$, consider a pair of operators

$$M_w = \partial_w - \lambda G g^{\tilde{\sigma} z}\partial_{\tilde{\sigma}} - \frac{\lambda}{\mathrm{i}\hbar}\partial_w\Theta * = \ell_w - \frac{\lambda}{\mathrm{i}\hbar}\partial_w\Theta *$$

$$M_z = \partial_z + \lambda G g^{\tilde{\sigma} w}\partial_{\tilde{\sigma}} - \frac{\lambda}{\mathrm{i}\hbar}\partial_z\Theta * = \ell_z - \frac{\lambda}{\mathrm{i}\hbar}\partial_z\Theta *, \qquad \text{for} \quad \lambda \in \boldsymbol{CP}^1 - \{\infty\},$$

and, respectively,

$$\underline{M_w} = \frac{1}{\zeta}\partial_w - G g^{\tilde{\sigma} z}\partial_{\tilde{\sigma}} - \frac{1}{\mathrm{i}\hbar}\partial_w\Theta * = \underline{\ell_w} - \frac{1}{\mathrm{i}\hbar}\partial_w\Theta *$$

$$\underline{M_z} = \frac{1}{\zeta}\partial_z + G g^{\tilde{\sigma} w}\partial_{\tilde{\sigma}} - \frac{1}{\mathrm{i}\hbar}\partial_z\Theta * = \underline{\ell_z} - \frac{1}{\mathrm{i}\hbar}\partial_z\Theta *, \qquad \text{for} \quad \zeta \in \boldsymbol{CP}^1 - \{0\}.$$

Then one has

$$\epsilon^{\alpha\beta} M_\alpha M_\beta = \frac{\lambda^2}{\mathrm{i}\hbar} G\left[g^{\tilde{\beta}\alpha}\partial_{\tilde{\beta}}\partial_\alpha\Theta + \frac{1}{\mathrm{i}\hbar G}\epsilon^{\alpha\beta}\partial_\alpha\Theta * \partial_\beta\Theta\right].$$

Thus for each $\lambda \in \boldsymbol{CP}^1 - \{\infty\}$ this commutator vanishes iff Θ satisfies ME. Analogously for $\zeta \in \boldsymbol{CP}^1 - \{0\}$ $[\underline{M_w}, \underline{M_z}] = 0$ iff Θ satisfies the master equation.

If Θ is any solution of ME then Frobenius integrability conditions are satisfied and one can find a solution of the linear system

$$M_w\Psi(\lambda) = 0, \qquad M_z\Psi(\lambda) = 0 \tag{2.10}$$

where $\Psi(\lambda) \equiv \Psi(t, \hbar; w, z, \tilde{w}, \tilde{z}, \lambda) \in \mathcal{A}$. In particular, this solution is analytic in λ in some neighbourhood of $0 \in \boldsymbol{CP}^1$. We will construct such a solution from conserved charges.

Let $\eta_{(k)}$ $k = 0, 1, 2, \ldots$ denote conserved charges defined by recursion relations (2.7). For $\lambda \in \boldsymbol{CP}^1 - \{\infty\}$ we define

$$\Psi(\lambda) := \sum_{k=0}^{\infty}\lambda^k\eta_{(k)}(t, \hbar; w, z, \tilde{w}, \tilde{z}). \tag{2.11}$$

The conserved charges can be chosen such that the radius of convergence is greater than zero. As all $\eta_{(k)}$ satisfy (2.7) thus above $\Psi(\lambda)$ satisfies (2.10).

By a *fundamental solution* of the system (2.10) we mean a solution with value in the group $\mathrm{e}^{\mathcal{Q}}$. Taking appropriate solution $\eta_{(0)}$ of (2.5) we get $\Psi(\lambda)$ with free element equal to 1. In particular, if we take $\eta_{(0)} = 1$ then the recursion relation gives $\eta_{(1)} = \frac{1}{\mathrm{i}\hbar}\Theta$.

Two fundamental solutions $\Psi_1(\lambda)$ and $\Psi_2(\lambda)$ differ only by a twistor function, i.e., $\Psi_1(\lambda) = \Psi_2(\lambda) * \mathcal{H}$ where $\mathcal{H} : \mathcal{F} \to \mathrm{e}^{\mathcal{Q}}$ is constant along each twistor surface $\ell_\alpha\mathcal{H} = 0$, $\alpha = w, z$.

For $\zeta \in \boldsymbol{CP}^1 - \{0\}$, using another sequence of conserved charges $\{\eta'_{(k)}\}_{k=0}^{\infty}$, one can construct a fundamental solution of the system

$$\underline{M_w}\underline{\Psi}(\zeta) = 0, \qquad \underline{M_z}\underline{\Psi}(\zeta) = 0, \qquad \underline{\Psi}(\zeta) = \eta'_{(0)} + \sum_{k=1}^{\infty}\left(\frac{1}{\zeta}\right)^k\eta'_{(k)}. \tag{2.12}$$

This time $\{\eta'_{(k)}\}_{k=0}^{\infty}$ is a sequence for which $\eta'_{(k)} = \widetilde{\mathcal{R}}\eta'_{(k+1)}$, $k \geqslant 1$. The operator $\widetilde{\mathcal{R}}$ is defined by (2.7). Moreover $\eta'_{(0)}$ fulfils additionally $\mathcal{D}^\alpha\eta'_{(0)} = 0$ which are the same as those for J^{-1} in Yang's equation (1.11) (cf Mason and Woodhouse (1996)).

On the overlap of domains, for $\lambda = \zeta \in \boldsymbol{CP}^1 - \{0, \infty\}$, $\Psi(\lambda) = \underline{\Psi}(\lambda) * \mathcal{H}$ where $\mathcal{H}$ is a twistor function defined uniquely by $\Psi(\lambda)$ and $\underline{\Psi}(\lambda)$. As it takes values in the group $\mathrm{e}^{\mathcal{Q}}$ and twistor space may be covered only by those two neighbourhoods the knowledge of this function is sufficient to recover a bundle over $\mathcal{PT}$ with $\mathcal{H}$ as a transition function. In this way, each solution of (ME) corresponds to one bundle over the space $\mathcal{PT}$.

2.5. Dressing operator

The hidden symmetries, i.e., elements of the space $T\mathcal{W}$ of solutions to equation (2.1) define a second Lax pair

$$\Phi(\lambda) := \sum_{n=0}^{\infty} \lambda^n \phi_{(n)}, \qquad \lambda \in \boldsymbol{CP}^1 - \{\infty\}. \tag{2.13}$$

As all $\phi_{(n)}, n = 0, 1, \ldots$ satisfy LME (2.1) $\Phi(\lambda)$ satisfies the system

$$\partial_\alpha \Phi(\lambda) - \lambda G \epsilon_{\alpha\beta} \mathcal{L}^\beta \Phi(\lambda) = 0, \qquad \lambda \in \boldsymbol{CP}^1 - \{\infty\}, \tag{2.14}$$

which is a Lax pair for ME.

Correspondingly, in the neighbourhood of infinity $\zeta \in \boldsymbol{CP}^1 - \{0\}$

$$\underline{\Phi}(\zeta) = \sum_{n=0}^{\infty} \left(\frac{1}{\zeta}\right)^n \phi_{(n)}, \qquad \zeta \in \boldsymbol{CP}^1 - \{0\} \tag{2.15}$$

then the spectral system is of the form

$$\frac{1}{\zeta} \partial_\alpha \underline{\Phi}(\zeta) - G \epsilon_{\alpha\beta} \mathcal{L}^\beta \underline{\Phi}(\zeta) = 0. \tag{2.16}$$

Let $F(\lambda) := F(t, \hbar; w, z, \tilde{w}, \tilde{z}, \lambda)$ be such that

$$\Phi(\lambda) = \Psi(\lambda) * F(\lambda) * \Psi^{-1}(\lambda). \tag{2.17}$$

Such $F(\lambda)$ exists and is uniquely defined by $\Psi(\lambda)$ and $\Phi(\lambda)$. Moreover, as $\Psi(\lambda)$ and $\Phi(\lambda)$ fulfil (2.10) and (2.14) respectively, $F(\lambda)$ has to be constant along each twistor surface, i.e., it depends only on (P^w, P^z, λ).

The definition (2.17) describing $F(\lambda)$ constitutes $\Psi(\lambda)$ as a *dressing operator* for a linear system (2.14).

2.6. Algebra of hidden symmetries

Consider a superposition of solutions to the linearized master equation (2.1), written in terms of the above-defined dressing operator

$$\delta_{(F\underline{F})}\Theta = \frac{1}{2\pi \mathrm{i}} \oint_\gamma \frac{\mathrm{d}\lambda}{\lambda^2} (-\Psi(\lambda) * F(\lambda) * \Psi^{-1}(\lambda) + \underline{\Psi}(\lambda) * \underline{F}(\lambda) * \underline{\Psi}^{-1}(\lambda)) \tag{2.18}$$

where

$$F(\lambda) = F(t, \hbar; \widetilde{P^w}, \widetilde{P^z}, \lambda), \qquad \underline{F}(\lambda) = \underline{F}(t, \hbar; \underline{P^w}, \underline{P^z}, \lambda), \tag{2.18}$$

(cf Park (1990, 1992), Takasaki (1990)). The contour γ in (2.18) is the boundary of a domain containing $\lambda = 0$ and it does not cross any singularity of integrated functions.

To find an algebra of hidden symmetries consider a *commutator*

$$\left[\delta_{(F_1\underline{F_1})}, \delta_{(F_2\underline{F_2})}\right]\Theta = \delta_{(F_1\underline{F_1})}\left(\Theta + \delta_{(F_2\underline{F_2})}\Theta\right) - \delta_{(F_1\underline{F_1})}\Theta - \delta_{(F_2\underline{F_2})}\left(\Theta + \delta_{(F_1\underline{F_1})}\Theta\right) + \delta_{(F_2\underline{F_2})}\Theta. \tag{2.19}$$

The following theorem holds (cf Takasaki (1990), Park (1992), Dunajski and Mason (2000)).

The hidden symmetries of ME constitute the algebra

$$\left[\delta_{(F_1\underline{F_1})}, \delta_{(F_2\underline{F_2})}\right]\Theta = \delta_{(\{F_1,F_2\}\{\underline{F_1},\underline{F_2}\})}\Theta. \tag{2.20}$$

The proof can be found in Przanowski *et al* (2001b), Formański (2004).

3. Integrability of ME

3.1. The homogenous Hilbert problem for formal power series

The Hilbert problem for formal power series can be defined in the similar form as in the case of vector functions. The latter case can be found in the monographs (Muscheliszwili 1962, Pogorzelski 1966). We will show the existence theorem in the first case.

Let L be a smooth contour. Let S^+ denote the interior of L and let $0 \in S^+$. By S^- we denote the exterior of L, i.e. $S^- := \boldsymbol{CP}^1 - S^+ - L$.

Let the formal power series

$$\Phi(t,\hbar;\lambda) = \sum_{m=0}^{\infty}\sum_{k=-m}^{\infty} t^m\hbar^k\Phi_{m,k}(\lambda)$$

be such that all the functions $\Phi_{m,k}(\lambda)$ are *sectionally holomorphic* which means that each $\Phi_{m,k}(\lambda)$ is holomorphic on S^+ and S^-. The formal power series is said to have a *finite degree at infinity* if for each function $\Phi_{m,k}(\lambda)$ there exist $c_{m,k} \in \boldsymbol{Z}$ such that $\lim_{|\lambda|\to\infty}\frac{|\Phi_{m,k}(\lambda)|}{|\lambda|^{c_{m,k}}} = 0$. In case $c_{m,k} > 0$ in the neighbourhood of infinity we can write

$$\Phi_{m,k}(\lambda) = \gamma_{m,k}(\lambda) + O\left(\frac{1}{\lambda}\right) \qquad \text{where} \quad \gamma_{m,k}(\lambda) \text{ is a polynomial.}$$

For $c_{m,k} < 0$ we have $\gamma_{m,k}(\lambda) = 0$ and for $c_{m,k} = 0$ the γ are constant. The formal power series $\gamma(t,\hbar;\lambda) = \sum_{m=0}^{\infty}\sum_{k=-m}^{\infty} t^m\hbar^k\gamma_{m,k}(\lambda)$ will be called the *principal part at infinity* of the series $\Phi(t,\hbar;\lambda)$. It is said that the series $\Phi(t,\hbar;\tau)\, \tau \in L$ satisfies on L the *Hölder condition* $H(\alpha)$, $0 < \alpha \leqslant 1$ if there exist constants $A_{m,k}$ such that $\forall\tau_1,\tau_2 \in L$ $|\Phi_{m,k}(\tau_2) - \Phi_{m,k}(\tau_1)| \leqslant A_{m,k}|\tau_2) - \tau_1|^{\alpha}$.

The homogeneous Hilbert problem can be formulated as follows. Suppose that we are given an element $G(t,\hbar;\xi)$ of the group $\mathrm{e}^{\mathcal{Q}}$, defined on L and satisfying the Hölder condition on L. Find a sectionally holomorphic formal power series $\Phi(t,\hbar;\lambda)$ having finite degree at infinity, continuous on L and satisfying the boundary condition

$$\Phi^+(t,\hbar;\xi) = \Phi^-(t,\hbar;\xi) * G(t,\hbar;\xi) \qquad \xi \in L \tag{3.1}$$

$\Phi^+(t,\hbar;\xi)$ and $\Phi^-(t,\hbar;\xi)$ denote the limit values, i.e.

$$\Phi^+(t,\hbar;\xi) = \lim_{\lambda\to\xi}\Phi(t,\hbar;\lambda) \qquad \text{for} \quad \lambda \in S^+$$

$$\Phi^-(t,\hbar;\xi) = \lim_{\lambda\to\xi}\Phi(t,\hbar;\lambda) \qquad \text{for} \quad \lambda \in S^-.$$

We will seek a solution of this problem in the class of formal power series satisfying Hölder condition on L.

In the case of finite groups this problem is solved by the *Birkhoff factorization theorem* (Birkhoff 1913, Mason and Woodhouse 1996).

Since $\Phi^+(t,\hbar;\xi)$ is the limit value of $\Phi(t,\hbar;\lambda)$ holomorphic in S^+, from the Cauchy theorem we find

$$0 = \frac{1}{2\pi\mathrm{i}}\int_L \frac{\Phi^+(t,\hbar;\tau)}{\tau-\lambda}\,\mathrm{d}\tau \qquad \lambda \in S^-.$$

By the Plemelj formula (Plemelj 1908), in the limit $\lambda \to \xi \in L$ one gets

$$-\frac{1}{2}\Phi^{+}(t,\hbar;\xi) + \frac{1}{2\pi \mathrm{i}}\int_L \frac{\Phi^{+}(t,\hbar;\tau)}{\tau-\xi}\,\mathrm{d}\tau = 0 \tag{3.2}$$

where the integral above is taken in the sense of principal value. The equation (3.2) is an integral equation for $\Phi^{+}(t,\hbar;\xi)$.

Analogously, the Cauchy theorem guarantees that $\Phi^{-}(t,\hbar;\xi)$ satisfies the integral equation

$$\frac{1}{2}\Phi^{-}(t,\hbar;\xi) + \frac{1}{2\pi \mathrm{i}}\int_L \frac{\Phi^{-}(t,\hbar;\tau)}{\tau-\xi}\,\mathrm{d}\tau = \gamma(t,\hbar;\xi) \tag{3.3}$$

where $\gamma(t,\hbar;\lambda)$ is a principal value at infinity of the series $\Phi(t,\hbar;\lambda)$.

Equations (3.2) and (3.3) and the condition $\Phi^{+}(t,\hbar;\xi) = \Phi^{-}(t,\hbar;\xi) * G(t,\hbar;\xi)$ imply the Fredholm integral equation with a non-singular kernel

$$\Phi^{-}(t,\hbar;\xi) - \frac{1}{2\pi \mathrm{i}}\int_L \Phi^{-}(t,\hbar;\tau) * \frac{G(t,\hbar;\tau) * G^{-1}(t,\hbar;\xi) - 1}{\tau-\xi}\,\mathrm{d}\tau = \gamma(t,\hbar;\xi). \tag{3.4}$$

Summarizing, the existence of a solution of the homogeneous Hilbert problem (3.1) implies that the limiting value $\Phi^{-}(t,\hbar;\xi)$ satisfies (3.4). The converse may not be true, as the solution of (3.4) has to satisfy additionally (3.2) and (3.3).

To answer under what condition the solution of (3.4) defines a sectionally holomorphic solution to the homogeneous Hilbert problem, consider

$$\Psi(t,\hbar;\lambda) = \begin{cases} \dfrac{1}{2\pi \mathrm{i}}\displaystyle\int_L \frac{\Phi^{-}(t,\hbar;\tau)}{\tau-\lambda}\,\mathrm{d}\tau - \gamma(t,\hbar;\lambda) & \text{for} \quad \lambda \in S^{+} \\ \dfrac{1}{2\pi \mathrm{i}}\displaystyle\int_L \frac{\Phi^{-}(t,\hbar;\tau) * G(t,\hbar;\tau)}{\tau-\lambda}\,\mathrm{d}\tau & \text{for} \quad \lambda \in S^{-}. \end{cases}$$

Thus $\Psi(t,\hbar;\lambda)$ is sectionally holomorphic and vanishes at infinity. The Plemelj theorem gives

$$\Psi^{+}(t,\hbar;\xi) = \frac{1}{2}\Phi^{-}(t,\hbar;\xi) + \frac{1}{2\pi \mathrm{i}}\int_L \frac{\Phi^{-}(t,\hbar;\tau)}{\tau-\xi}\,\mathrm{d}\tau - \gamma(t,\hbar;\xi)$$

$$\Psi^{-}(t,\hbar;\xi) = -\frac{1}{2}\Phi^{-}(t,\hbar;\xi) * G(t,\hbar;\xi) + \frac{1}{2\pi \mathrm{i}}\int_L \frac{\Phi^{+}(t,\hbar;\tau) * G(t,\hbar;\tau)}{\tau-\xi}\,\mathrm{d}\tau.$$

As is seen from (3.1), equation (3.2) is equivalent to vanishing of $\Psi^{-}(t,\hbar;\xi)$ on L. Equation (3.3) gives $\Psi^{+}(t,\hbar;\xi) = 0$ on L. Those two conditions and the fact that $\Psi(t,\hbar;\lambda)$ is holomorphic on S^{+} and S^{-} lead to $\Psi(t,\hbar;\lambda) \equiv 0$. The integral equation (3.4) is a condition

$$\Psi^{+}(t,\hbar;\xi) = \Psi^{-}(t,\hbar;\xi) * G^{-1}(t,\hbar;\xi) \qquad \xi \in L. \tag{3.5}$$

The problem of finding $\Psi(t,\hbar;\lambda)$ sectionally holomorphic, vanishing at infinity, satisfying boundary condition (3.5) on L is called *the accompanying problem* of the problem (3.1). Analogously, this problem implies the integral equation for a limit value $\Psi^{+}(t,\hbar;\xi)$

$$\Psi^{+}(t,\hbar;\xi) + \frac{1}{2\pi \mathrm{i}}\int_L \Psi^{+}(t,\hbar;\tau) * \frac{G(t,\hbar;\tau) * G^{-1}(t,\hbar;\xi) - 1}{\tau-\xi}\,\mathrm{d}\tau = 0. \tag{3.6}$$

As is seen from the above, the solution of integral equation (3.4) defines the solution of original homogenous Hilbert problem (3.1) iff the only solution of the accompanying problem is the trivial one $\Psi(t,\hbar;\xi) \equiv 0$.

Thus, in order to prove the existence of the solution of the problem (3.1) we need to prove that equation (3.6) has only the trivial solution and that there exists a solution of the (3.4).

To simplify the notation, we will denote

$$\Phi(t,\hbar;\lambda)=\sum_{m=0}^{\infty}t^m\Phi_m(\hbar;\lambda)\qquad\text{where}\quad \Phi_m(\hbar;\lambda)=\sum_{k=-m}^{\infty}\hbar^k\Phi_{m,k}(\lambda).$$

First observe that the free element of $G(t,\hbar;\tau)*G^{-1}(t,\hbar;\xi)-1$ vanishes, so we can write

$$G(t,\hbar;\tau)*G^{-1}(t,\hbar;\xi)-1=\sum_{n=1}^{\infty}t^nF_n(\hbar;\tau,\xi).$$

Inserting into (3.6) one gets

$$\sum_{m=0}^{\infty}t^m\Psi_m^+(\hbar;\xi)=-\frac{1}{2\pi\mathrm{i}}\int_L\sum_{s=0}^{\infty}t^s\Psi_s^+(\hbar;\tau)*\frac{\sum_{n=1}^{\infty}t^nF_n(\hbar;\tau,\xi)}{\tau-\xi}\,\mathrm{d}\tau.$$

This equation can be solved iteratively

$$\Psi_0^+(\hbar;\xi)=0$$

$$\Psi_m^+(\hbar;\xi)=-\frac{1}{2\pi\mathrm{i}}\int_L\frac{\sum_{j=0}^{m-1}\Psi_j^+(\hbar;\xi)*F_{m-j}(\hbar;\tau,\xi)}{\tau-\xi}\,\mathrm{d}\tau,\qquad m\geqslant 1.$$

Thus the only solution of (3.6) is $\Psi^+(t,\hbar;\xi)=0$. Consequently, each solution of (3.4) defines a solution of Hilbert problem

$$\Phi(t,\hbar;\lambda)=\begin{cases}\dfrac{1}{2\pi\mathrm{i}}\displaystyle\int_L\Phi^-(t,\hbar;\tau)*\frac{G(t,\hbar;\tau)}{\tau-\lambda}\,\mathrm{d}\tau & \text{dla}\quad \lambda\in S^+\\[2ex] -\dfrac{1}{2\pi i}\displaystyle\int_L\frac{\Phi^-(t,\hbar;\tau)}{\tau-\lambda}\,\mathrm{d}\tau+\gamma(t,\hbar;\lambda) & \text{dla}\quad \lambda\in S^-.\end{cases}$$

Equation (3.4) takes the form

$$\sum_{m=0}^{\infty}t^m\Phi_m^-(\hbar;\xi)=-\frac{1}{2\pi\mathrm{i}}\int_L\sum_{s=0}^{\infty}t^s\Phi_s^-(\hbar;\tau)*\frac{\sum_{n=1}^{\infty}t^nF_n(\hbar;\tau,\xi)}{\tau-\xi}\,\mathrm{d}\tau+\sum_{m=0}^{\infty}t^m\gamma_m(\hbar;\xi)$$

and it can be solved iteratively

$$\Phi_m^-(\hbar;\xi)=\gamma_m(\hbar;\xi)+\frac{1}{2\pi\mathrm{i}}\int_L\frac{\sum_{j=0}^{m-1}\Phi_j^-(\hbar;\xi)*F_{m-j}(\hbar;\tau,\xi)}{\tau-\xi}\,\mathrm{d}\tau\quad m=0,1,2,\ldots.$$

Note that the solution of (3.4), takes the value in a group $\mathrm{e}^{\mathcal{Q}}$ iff $\gamma(t,\hbar;\lambda)\in\mathrm{e}^{\mathcal{Q}}$.

Some remarks on Riemann–Hilbert problem for ∗-algebra can also be found in Takasaki (1994), Strachan (1997).

3.2. Inverse Penrose–Ward transform

In this section, we will show the correspondence between holomorphic formal bundles over twistor space $\mathcal{PT}$ and solutions of master equation (1.16).

As it was shown the manifold $\mathcal{PT}$ is covered by two coordinate neighbourhoods $(V,(P^w,P^z,\lambda))$, $\lambda\in\boldsymbol{CP}^1-\{\infty\}$ and $(\underline{V},(\underline{P^w},\underline{P^z},\zeta))$, $\zeta\in\boldsymbol{CP}^1-\{0\}$.

Each holomorphic formal bundle over $\mathcal{PT}$ is characterized by a transition function $H(t,\hbar;P^w,P^z,\lambda):V\cap\underline{V}\to\mathrm{e}^{\mathcal{Q}}$, i.e.,

$$H(t,\hbar;\widetilde{P^w},\widetilde{P^z},\lambda)=1+\sum_{m=1}^{\infty}\sum_{k=-m}^{\infty}t^m\hbar^kH_{m,k}(P^w,P^z,\lambda)$$

with $H_{m,k}(P^w, P^z, \lambda)$ being holomorphic. The pull-back of the series by the map $p : \mathcal{M} \times \boldsymbol{CP}^1 \to \mathcal{PT}$ gives on the correspondence space $\mathcal{F} = \mathcal{M} \times \boldsymbol{CP}^1$ the series $\mathcal{H} = p^* H$ constant along each twistor surface

$$\ell_w \mathcal{H} = 0, \qquad \ell_z \mathcal{H} = 0. \tag{3.7}$$

Briefly we will write $\mathcal{H}(t, \hbar; \lambda) \equiv \mathcal{H}(t, \hbar; w, z, \tilde{w}, \tilde{z}, \lambda)$.

As H is a transition function, $\mathcal{H}(t, \hbar; \lambda)$ can be factorized as

$$\Psi(t, \hbar; \lambda) = \underline{\Psi}(t, \hbar; \lambda) * \mathcal{H}(t, \hbar; \lambda) \qquad \text{for} \quad \lambda \in \boldsymbol{CP}^1 - \{0, \infty\} \tag{3.8}$$

where $\Psi(t, \hbar; \lambda)$ is holomorphic everywhere apart from $\lambda = \infty$ and $\underline{\Psi}(t, \hbar; \lambda)$ is holomorphic everywhere apart from $\lambda = 0$ and both series take values in $\mathrm{e}^{\mathcal{Q}}$.

The problem of such factorization, known as the Riemann–Hilbert problem, reduces to the previously discussed homogenous Hilbert problem.

Indeed, let L be a smooth contour on $\boldsymbol{CP}^1$ (for example an equator). One can find $\Psi(t, \hbar; \lambda)$ holomorphic on S^+ (we use the same notation as in the previous sections), $\underline{\Psi}(t, \hbar; \lambda)$ holomorphic on S^- and continuous on $S^+ \cup L$ and $S^- \cup L$, respectively. On L they satisfy the condition

$$\widetilde{\Psi}^+(t, \hbar; \xi) = \underline{\Psi}^-(t, \hbar; \xi) * \mathcal{H}(t, \hbar; \xi) \qquad \xi \in L.$$

The series Ψ and $\underline{\Psi}$ can be analytically continued onto $S^- - \{\infty\}$ and $S^+ - \{0\}$, respectively, by

$$\begin{aligned} &\text{for} \quad \lambda \in S^- - \{\infty\} \qquad \Psi(t, \hbar; \lambda) := \underline{\Psi}(t, \hbar; \lambda) * \mathcal{H}(t, \hbar; \lambda) \\ &\text{for} \quad \lambda \in S^+ - \{0\} \qquad \underline{\Psi}(t, \hbar; \lambda) := \Psi(t, \hbar; \lambda) * \mathcal{H}^{-1}(t, \hbar; \lambda). \end{aligned} \tag{3.9}$$

In this way, we obtain $\Psi(t, \hbar; \lambda)$ defined in each finite point of the complex plane and sectionally holomorphic on S^+, S^-, satisfying on L the condition $\Psi^+(t, \hbar; \xi) = \Psi^-(t, \hbar; \xi)$. This means that such a $\Psi(t, \hbar; \lambda)$ is holomorphic on the whole complex plane, as desired. Analogously $\underline{\Psi}(t, \hbar; \lambda)$ is holomorphic on $\boldsymbol{C} - \{0\}$. From the definition (3.9) the factorization (3.8) holds.

Suppose that we are given $\Psi(t, \hbar; \lambda)$ and $\underline{\Psi}(t, \hbar; \lambda)$ defined by (3.8). Thus from (3.7) one gets

$$\ell_\alpha [\underline{\Psi}^{-1}(t, \hbar; \lambda) * \Psi(t, \hbar; \lambda)] = 0, \qquad \alpha = w, z.$$

and

$$\ell_\alpha \Psi(t, \hbar; \lambda) * \Psi^{-1}(t, \hbar; \lambda) = \ell_\alpha \underline{\Psi}(t, \hbar; \lambda) * \underline{\Psi}^{-1}(t, \hbar; \lambda).$$

The LHS is holomorphic everywhere apart from $\lambda = \infty$, while RHS is holomorphic everywhere apart from $\lambda = 0$ and at infinity it may have only a first-order pole. Thus from the *Liouville theorem* they both are linear with respect to λ, i.e.,

$$\begin{aligned} \ell_\alpha \Psi(t, \hbar; \lambda) &= \frac{1}{\mathrm{i}\hbar}(-A_\alpha + \lambda \epsilon_{\alpha\beta} G g^{\tilde{\sigma}\beta} A_{\tilde{\sigma}}) * \Psi(t, \hbar; \lambda) \\ \ell_\alpha \underline{\Psi}(t, \hbar; \lambda) &= \frac{1}{\mathrm{i}\hbar}(-A_\alpha + \lambda \epsilon_{\alpha\beta} G g^{\tilde{\sigma}\beta} A_{\tilde{\sigma}}) * \underline{\Psi}(t, \hbar; \lambda) \end{aligned}$$

where $A_\alpha = A_\alpha(t, \hbar), \alpha = w, z, A_{\tilde{\sigma}} = A_{\tilde{\sigma}}(t, \hbar), \tilde{\sigma} =, \tilde{w}, \tilde{z}$ do not depend on λ.

The $A_\alpha, A_{\tilde{\sigma}}$ satisfy the SDYM equations (1.8), (1.9) and (1.10). This is easily seen from the fact that the vector fields ℓ_w, ℓ_z commute for each value of λ. Thus

$$\begin{aligned} 0 &= \epsilon^{\beta\alpha} \ell_\beta \ell_\alpha \Psi(\lambda) = \epsilon^{\beta\alpha}(\partial_\beta - \lambda G \epsilon_{\beta\delta} g^{\tilde{\sigma}\delta} \partial_{\tilde{\sigma}}) \left[\left(\frac{-1}{\mathrm{i}\hbar} A_\alpha + \frac{1}{\mathrm{i}\hbar} \lambda G \epsilon_{\alpha\sigma} g^{\tilde{\rho}\sigma} A_{\tilde{\rho}} \right) * \Psi \right] \\ &= \frac{1}{\mathrm{i}\hbar} \left[-\epsilon^{\beta\alpha} \partial_\beta A_\alpha + \frac{1}{\mathrm{i}\hbar} \epsilon^{\beta\alpha} A_\alpha * A_\beta \right] + \frac{\lambda}{\mathrm{i}\hbar} G g^{\tilde{\sigma}\alpha} \left[\partial_{\tilde{\sigma}} A_\alpha - \partial_\alpha A_{\tilde{\sigma}} + \frac{1}{\mathrm{i}\hbar}(A_{\tilde{\sigma}} * A_\alpha - A_\alpha * A_{\tilde{\sigma}}) \right] \\ &\quad + \frac{\lambda^2}{\mathrm{i}\hbar} \frac{G}{\tilde{G}} \epsilon^{\tilde{\rho}\tilde{\sigma}} \left[\partial_{\tilde{\rho}} A_{\tilde{\sigma}} + \frac{1}{\mathrm{i}\hbar} A_{\tilde{\rho}} * A_{\tilde{\sigma}} \right]. \end{aligned}$$

Appropriate terms at $\lambda^0, \lambda^1, \lambda^2$ give SDYM: (1.8), (1.10) and (1.9), respectively.

Let us note that the factorization (3.8) does not define the series $\Psi(t,\hbar;\lambda)$, $\underline{\Psi}(t,\hbar;\lambda)$ uniquely. They can be simultaneously multiplied by $a(t,\hbar) \in \mathrm{e}^{\mathcal{Q}}$, independent of λ. This implies the gauge freedom for SDYM potential.

Thus we can choose such $\Psi(t,\hbar;\lambda)$, $\underline{\Psi}(t,\hbar;\lambda)$ that $A_\alpha = 0$. Then (compare with (2.10))

$$\ell_\alpha \Psi(t,\hbar;\lambda) * \Psi^{-1}(t,\hbar;\lambda) = \frac{1}{\mathrm{i}\hbar}\partial_\alpha \Theta \tag{3.10}$$

where Θ is a solution of the master equation (1.16).

4. Conclusions

In this work, we have found the evidence of integrability of $*$-SDYM equations. This evidence follows from

- the existence of infinite number of conservation laws,
- the existence of Lax pair,
- the one-to-one correspondence between solutions of $*$-SDYM equations and formal holomorphic bundles over $\mathcal{PT}$ with structure group $\mathrm{e}^{\mathcal{Q}}$.
- the existence of solution to the Riemann–Hilbert problem what gives rise to an algebraic method of generating solutions to (ME).

In the second part of this paper some examples of reductions of $*$-SDYM to other integrable systems, such as $SU(N)$-SDYM equations, $SU(N)$ chiral equations and heavenly equations, will be given. We also find a sequence of $SU(N)$ chiral fields tending to the heavenly space when $N \to \infty$.

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